WHEN DOES A SCHRÖDINGER HEAT EQUATION PERMIT POSITIVE SOLUTIONS

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ABSTRACT. We introduce some new classes of time dependent functions whose defining properties take into account of oscillations around singularities. We study properties of solutions to the heat equation with coefficients in these classes which are much more singular than those allowed under the current theory. In the case of L^2 potentials and L^2 solutions, we give a characterization of potentials which allow the Schrödinger heat equation to have a positive solution. This provides a new result on the long running problem of identifying potentials permitting a positive solution to the Schrödinger equation.

We also establish a nearly necessary and sufficient condition on certain sign changing potentials such that the corresponding heat kernel has Gaussian upper and lower bound.

Some applications to the Navier-Stokes equations are given. In particular, we derive a new type of a priori estimate for solutions of Navier-Stokes equations. The point is that the gap between this estimate and a sufficient condition for all time smoothness of the solution is logarithmic .

CONTENTS

1. INTRODUCTION

In the first part of the paper we would like to study the heat equation with a singular L^2 potential $V = V(x,t)$, i.e.

$$
\begin{cases} \Delta u + Vu - u_t = 0, \quad \text{in} \quad \mathbf{R}^n \times (0,\infty), \quad V \in L^2(\mathbf{R}^n \times (0,\infty)),\ n \geq 3, \\ u(x,0) = u_0(x), \quad x \in \mathbf{R}^n, \quad u_0 \in L^2(\mathbf{R}^n). \end{cases} \tag{1.1}
$$

Since we are only concerned with local regularity issue in this paper, we will always assume that V is zero outside of a cylinder in space time: $B(0,R_0)\times[0,T_0]$, unless stated otherwise. Here R_0 and T_0 are fixed positive number. The L^2 condition on the potential V is modeled after the three dimensional vorticity equation derived from the Navier-Stokes equation. There the potential V is in fact the gradient of the velocity which is known to be in L^2.

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The unknown function u in (1.1) corresponds to the vorticity which is also known to be a L^2 function.

We will use the following definition of weak solutions.

Definition 1.1. Let $T > 0$. We say that $u \in L^1_{loc}(\mathbf{R}^n \times (0, , T))$ is a solution to (1.1) if $V(.)u(., t) \in L^1_{loc}(\mathbf{R}^n \times (0, T))$ and

$$\int_{\mathbf{R}^n} u_0(x)\phi(x)dx + \int_0^T \int_{\mathbf{R}^n} u\phi_t dxdt + \int_0^T \int_D u\Delta\phi dxdt + \int_0^T \int_{\mathbf{R}^n} Vu\phi dxdt = 0$$

for all smooth, compactly supported ϕ vanishing on $\mathbf{R}^n \times \{T\}$.

It is well known that L^2 potentials in general are too singular to allow weak solutions of (1.1) to be bounded or unique. Therefor further assumptions must be imposed in order to establish a regularity theory. The classical condition on the potential V for Hölder continuity and uniqueness of weak solutions is that $V \in L^{p,q}_{loc}$ with $\frac{n}{p} + \frac{2}{q} < 2$. This condition is sharp in general since one can easily construct a counter example. For instance for $V = a/|x|^2$ with $a > 0$, then there is no bounded positive solution to (1.1) (c.f. [BG]). In fact in that paper, it is shown that if a is sufficiently large, then even weak positive solutions can not exist. There is a long history of finding larger class of potentials such that some regularity of the weak solutions is possible. Among them is the Kato class, time independent or otherwise. Roughly speaking a function is in a Kato type class if the convolution of the absolute value of the function and the fundamental solution of Laplace or the heat equation is bounded. This class of functions are moderately more general than the standard $L^{p,q}$ class. However, it is still far from enough for applications in such places as the vorticity equation mentioned above. We refer the reader to the papers [AS], [Si], [Z], [LS] and reference therein for results in this direction. The main results there is the continuity of weak solutions with potentials in the Kato class. In addition, equation (1.1) with V in Morrey or Besov classes are also studied. However, these classes are essentially logarithmic improvements over the standard $L^{p,q}$ class. In the paper [St], K. Sturm proved Gaussian upper and lower bound for the fundamental solution when the potential belongs to a class of time independent, singular oscillating functions. His condition is on the L^1 bound of the fundamental solution of a slightly "larger" potential.

In this paper we introduce a new class of *time dependent* potentials which can be written as a nonlinear combination of derivatives of a function. The general idea of studying elliptic and parabolic equations with potentials as the spatial derivative of some functions is not new. This has been used in the classical books [LSU], [GT] and [Lieb]. Here we also allow the appearance of time derivative which can not be dominated by the Laplace operator. Another innovation is the use of a suitable combination of derivatives. The class we are going to define in section 2 essentially characterize all L^2 potentials which allow (1.1) to have positive L^2 solutions.

The question of whether the Laplace or the heat equation with a potential possesses a positive solution has been a long standing one. For the Laplace equation, when the potential has only mild singularity, i.e. in the Kato class, a satisfactory answer can be found in the Allegretto-Piepenbrink theory. See Theorem C.8.1 in the survey paper [Si]. Brezis and J. L. Lions (see [BG] p122) asked when (1.1) with more singular potential has a positive solutions. This problem was solved in [BG] when $V = a/|x|^2$ with $a > 0$. In the case of general time independent potentials $V \geq 0$, it was solved in [CM] and later generalized in [GZ]. However the case of time dependent or sign changing potentials is completely open.

One of the main results of the paper (Theorem 2.1) gives a solution of the problem with L^2 potentials. The main advantage of the new class of potentials is that it correctly captures the cancelation effect of sign changing functions. Moreover, we show in Theorem 2.2-3 below that, if we just narrow the class a little, then the weak fundamental solutions not only exist but also have Gaussian upper bound. A Gaussian lower bound is also established under further but necessary restrictions.

Some of the results of the paper can be generalized beyond L^2 potentials. However we will not seek full generalization this time.

Before proceeding further let us fix some notations and symbols, to which will refer the reader going over the rest of the paper.

Notations. We will use $\mathbf{R}^+$ to denote $(0, \infty)$. The letter C, c with or without index will denote generic positive constants whose value may change from line to line, unless specified otherwise. When we say a time dependent function is in L^2 we mean its square is integrable in $\mathbf{R}^n \times \mathbf{R}^+$. We use G_V to denote the fundamental solution of (1.1) if it exists. Please see the next section for its existence and uniqueness. The symbol G_0 will denote the fundamental solution of the heat equation free of potentials. Give $b > 0$, we will use g_b to denote a Gaussian with b as the exponential parameter, i.e.

$$g_b = g_b(x, t; y, s) = \frac{1}{(t-s)^{n/2}} e^{-b|x-y|^2/(t-s)}.$$

Given a L^1_{loc} function f in space time, we will use $g_b \star f(x, t)$ to denote

$$\int_0^t \int_{\mathbf{R}^n} g_b(x, t; y, s) f(y, s) dy ds.$$

When we say that G_V has Gaussian upper bound, we mean that exists $b > 0$ and $c > 0$ such that $G_V(x, t; y, s) \le c g_b(x, t; y, s)$. The same goes for the Gaussian lower bound.

When we say a function is a positive solution to (1.1) we mean it is a nonnegative weak solution which is not identically zero.

Here is the plan of the paper. In the next section we provide the definitions, statements and proofs of the main results. In section 3, we define another class of singular potentials, called heat bounded class. Some applications to the Navier-Stokes equation is given in Section 4.

2. SINGULAR POTENTIALS AS COMBINATIONS OF DERIVATIVES

2.1. **Definitions, Statements of Theorems.**

Definition 2.1. *Given two functions $V \in L^2(\mathbf{R}^n \times \mathbf{R}^+)$, $f \in L^1_{loc}(\mathbf{R}^n \times \mathbf{R}^+)$ and $\alpha > 0$, we say that*

$$V = \Delta f - \alpha |\nabla f|^2 - f_t,$$

if there exists sequences of functions $\{V_i\}$ and $\{f_i\}$ such that the following conditions hold for all $i = 1, 2...$:
(i). $V_i \in L^2(\mathbf{R}^n \times \mathbf{R}^+)$, $\Delta f_i \in L^2(\mathbf{R}^n \times \mathbf{R}^+)$, $\partial_t f_i \in L^2(\mathbf{R}^n \times \mathbf{R}^+)$, $f_i \in C(\mathbf{R}^n \times \mathbf{R}^+)$.
(ii). $V_i \to V$ strongly in $L^2(\mathbf{R}^n \times \mathbf{R}^+)$, $|V_i| \le |V_{i+1}|$, $f_i \to f$ a.e. and $f_i(x, 0) = f_{i+1}(x, 0)$.
(iii).

$$V_i = \Delta f_i - \alpha |\nabla f_i|^2 - \partial_t f_i.$$

Here we remark that we do not assume Δf, $|\nabla f|^2$ or $\partial_t f$ are in L^2 individually. This explains the lengthy appearance of the definition.

The main results of Section 2 are the next three theorems. The first one states a necessary and sufficient condition such that (1.1) possesses a positive solution. The second theorem establishes Gaussian upper and lower bound for the fundamental solutions of (1.1). The third theorem is an application of the second one in the more traditional setting of $L^{p,q}$ conditions on the potential. It will show that our conditions are genuinely much broader than the traditional ones.

It should be made clear that there is no claim on uniqueness in any of the theorems. In the absence of uniqueness how does one define the fundamental solution? This is possible due to the uniqueness of problem (1.1) when the potential V is truncated from above. This fact is proved in Proposition 2.1 below. Consequently we can state

Definition 2.2. *The fundamental solution G_V is defined as the pointwise limit of the (increasing) sequence of the fundamental solution G_{V_i} where $V_i = \min\{V,\ i\}$ with $i = 1, 2, ...$.*

We remark that G_V thus defined may be infinity somewhere or everywhere. However we will show that they have better behavior or even Gaussian bounds under further conditions.

Theorem 2.1. *(i). Suppose (1.1) with some $u_0 \ge 0$ has a positive solution. Then*

$$V = \Delta f - |\nabla f|^2 - \partial_t f,$$

with $e^{-f} \in L^2(\mathbf{R}^n \times \mathbf{R}^+)$. Moreover $f \in L^1_{loc}(\mathbf{R}^n \times \mathbf{R}^+)$ if $\ln u_0 \in L^1_{loc}(\mathbf{R}^n)$.

(ii). Suppose

$$V = \Delta f - |\nabla f|^2 - \partial_t f$$

for some f such that $e^{-f} \in L^2(\mathbf{R}^n \times \mathbf{R}^+)$. Then the equation in (1.1) has a positive L^2 solution for some $u_0 \in L^2(\mathbf{R}^n)$.

Theorem 2.2. *(i). Suppose*

$$V = \Delta f - \alpha|\nabla f|^2 - \partial_t f$$

for one given $\alpha > 1$ and $f \in L^\infty(\mathbf{R}^n \times \mathbf{R}^+)$. Then G_V has Gaussian upper bound in all space time.

(ii). Under the same assumption as in (i), if $g_{1/4} \star |\nabla f|^2 \in L^\infty(\mathbf{R}^n \times \mathbf{R}^+)$, then G_V has Gaussian lower bound in all space time.

(iii). Under the same assumption as in (i), suppose G_V has Gaussian lower bound in all space time. Then there exists $b > 0$ such that

$$g_b \star |\nabla f|^2 \in L^\infty(\mathbf{R}^n \times \mathbf{R}^+).$$

Remark 2.1. At the first glance, Theorem 2.1 may seem like a restatement of existence of positive solutions without much work. However Theorem 2.2 shows that if one just puts a little more restriction on the potential V, then the fundamental solution actually has a Gaussian upper bound. Under an additional but necessary assumption, a Gaussian lower bound also holds. Even the widely studied potential $a/|x|^2$ in $\mathbf{R}^n$ can be recast in the form of Theorem 1.1, as indicated in the following

Example 2.1. For a real number b, we write $f = b \ln r$ with $r = |x|$. Then direct calculation shows, for $r \neq 0$,

$$\frac{b(n-2-b)}{r^2} = \Delta f - |\nabla f|^2.$$

Let $a = b(n-2-b)$. Then it is clear that the range of a is $(-\infty, (n-2)^2/4]$. In this interval (1.1) with $V = a/|x|^2$ permits positive solutions. This recovers the existence part in the classical result [BG]. Highly singular, time dependent examples can be constructed by taking $f = \sin(\frac{1}{|x|-\sqrt{t}})$ e.g.

Moreover the corollaries below relate our class of potentials with the traditional "form bounded" or domination class (2.1), (see also [Si] below). In the difficult time dependent case, Corollary 1 shows that potentials permitting positive solutions, can be written as the sum of one form bounded potentials and the time derivative of a function almost bounded from above by a constant.

Remark 2.2. From the proof, it will be clear that under the assumption of part (i) of Theorem 2.2, one has

$$\int G_{\alpha V}(x,t;y,s)dy \leq C < \infty.$$

This is one of the main assumptions used by Sturm [St] in the time independent case (Theorem 4.12). If $\alpha = 1$, then the conclusion of Theorem 2.2 may not hold even for time independent potentials. See [St]. Also note that this theorem provides a nearly necessary and sufficient condition on certain sign changing potential such that the corresponding heat kernel has Gaussian upper and lower bound. The only "gap" in the condition is the difference in the parameters of the kernels $g_{1/4}$ and g_b. It is well know and easy to check if $|\nabla f| \in L^{p,q}$ with $\frac{n}{p} + \frac{2}{q} < 1$ and $f = 0$ outside a compact set, then $g_b \star |\nabla f|^2$ is a bounded function for all $b > 0$.

Corollary 1. *Let $V \in L^2(\mathbf{R}^n \times (0,\infty))$.*

(a). Suppose

$$\int_0^T \int V\phi^2 \leq \int_0^T \int |\nabla \phi|^2 + b \int_0^T \int \phi^2. \tag{2.1}$$

for all smooth, compactly supported function ϕ in $\mathbf{R}^n \times (0,T)$ and some $b > 0$ and $T > 0$. Then (1.1) has a positive solution when $u_0 \geq 0$ and moreover

$$V = \Delta f - |\nabla f|^2 - \partial_t f.$$

(b). Suppose $V = \Delta f - |\nabla f|^2$ then V is form bounded, i.e. it satisfies (2.1).

(c). Let V be a L^2 potential permitting positive L^2 solutions for (1.1), then V can be written as the sum of one form bounded potentials and the time derivative of a function almost bounded from above by a constant.

In the next corollary, we consider only time-independent, nonnegative potentials. Here the definition of $V = \Delta f - |\nabla f|^2$ is slightly different from that of Definition (1.1) since we do not have to worry about time derivatives. One interesting consequence is that these class of potentials is *exactly* the usual form boundedness potentials.

Corollary 2. *Suppose $0 \le V \in L^1(\mathbf{R}^n)$. Then the following statements are equivalent.*

(1). For some $f \in L^1_{loc}(\mathbf{R}^n)$ and a constant $b > 0$,

$$V = \Delta f - |\nabla f|^2 + b.$$

This mean there exist $V_j \in L^\infty$ such that $V_j \to V$ in $L^2(\mathbf{R}^1)$ and $V_j = \Delta f_j - |\nabla f_j|^2 + b$ for some $f_j \in W^{2,2}(\mathbf{R}^n)$, $j = 1, 2, \ldots.$

(2).

$$\int V\phi^2 dx \le \int |\nabla \phi|^2 dx + b \int \phi^2 dx.$$

for all smooth, compactly supported function ϕ in $\mathbf{R}^n$ and some $b \ge 0$.

Remark 2.3. Condition (2) in the above corollary just means that the bottom of the spectrum for the operator $-\Delta - V$ is finite. This condition is the same as those given in [CM] and [GZ].

It is a fact that most people feel more familiar with the case when the potential V is written as $L^{p,q}$ functions. Also there may be some inconvenience about the presence of the nonlinear term $|\nabla f|^2$ in the potential in Theorem 2.2. Therefore in our next theorem, we will use only $L^{p,q}$ conditions on f without nonlinear terms.

Theorem 2.3. *Suppose*

(a) $f \in L^\infty(\mathbf{R}^n \times \mathbf{R}^+)$;

(b) $f = 0$ outside a cylinder $B(0, R_0) \times [0, T_0]$, $R_0, T_0 > 0$;

(c) $|\nabla f| \in L^{p,q}(\mathbf{R}^n \times \mathbf{R}^+)$ with $\frac{n}{p} + \frac{2}{q} < 1$;

(d) $V = \Delta f - \partial_t f \in L^2(\mathbf{R}^n \times \mathbf{R}^+)$.

Then there exists a constant A_0 depending only on n, p, q such that the following statements hold, provided that

$$\| \, |\nabla f| \, \|_{L^{p,q}(\mathbf{R}^n \times \mathbf{R}^+)} < A_0.$$

(i) The kernel G_V has Gaussian upper and lower bound in all space time.

(ii) The kernel $G_{\partial_t f}$ has Gaussian upper and lower bound in all space time.

Remark 2.4. If V is independent of time, then Theorem 2.3. reduces to the well known classical fact:

if a potential V is the derivative a of a small $L^{n+\epsilon}$ function, then G_V has Gaussian upper and lower bound. (See [LS]) e.g.

In the time dependent case our result is genuinely new due to the presence of the term $\partial_t f$. Let us mention that some smallness condition on the potential is needed for the existence of Gaussian bounds for G_V. This is the case even for time independent, smooth potentials due to the possible presence of ground state.

2.2. **Preliminaries.** In order to prove the theorems we need to prove a proposition concerning the existence, uniqueness and maximum principle for solutions of (1.1) under the assumptions that V is bounded from above by a constant. The result is standard if one assumes that the gradient of solutions are L^2. However we only assume that solutions are L^2. Therefore a little extra work is needed.

Proposition 2.1. *Suppose that $V \in L^2(\mathbf{R}^n \times \mathbf{R}^+)$ and that $V \le b$ for a positive constant b. Then the following conclusions hold.*

(i). The only L^2 solution to the problem

$$\begin{cases} \Delta u + Vu - \partial_t u = 0, & \mathbf{R}^n \times (0,T), \ T>0, \\ u(x,0)=0, \end{cases}$$

is zero.

(ii). Let u be a solution to the problem in (i) such that $u(\cdot,t) \in L^1(\mathbf{R}^n)$, $u \in L^1(\mathbf{R}^n \times (0,T))$ and $Vu \in L^1(\mathbf{R}^n \times (0,T))$. Then u is identically zero.

(iii). Under the same assumptions as in (i), the following problem has a unique L^2 nonnegative solution.

$$\begin{cases} \Delta u + Vu - \partial_t u = 0, & \mathbf{R}^n \times (0,T), \ T>0, \\ u(\cdot,0)=u_0(\cdot) \ge 0, & u_0 \in L^2(\mathbf{R}^n). \end{cases}$$

(iv). Under the same assumptions as in (i), let u be a L^2 solution to the following problem

$$\begin{cases} \Delta u + Vu - \partial_t u = f, & \mathbf{R}^n \times (0,T), \ T>0, \\ u(\cdot,0)=0. \end{cases}$$

Here $f \le 0$ and $f \in L^1(\mathbf{R}^n \times (0,T))$. Then $u \ge 0$ in $\mathbf{R}^n \times (0,T)$.

Proof of (i). Let u be a L^2 solution to the problem in (i). Choose a standard mollifier ρ and define, for $j=1,2,...,$

$$u_j(x,t) = j^n \int \rho(j(x-y))u(y,t)dy \equiv \int \rho_j(x-y)u(y,t)dy.$$

Then ∇u_j and Δu_j exist in the classical sense. From the equation on u, it holds

$$\Delta u_j + \int \rho_j(x-y)V(y,t)u(y,t)dy - \partial_t u_j = 0,$$

where $\partial_t u_j$ is understood in the weak sense.

Given $\epsilon > 0$, we define

$$h_j = \sqrt{u_j^2 + \epsilon}.$$

Let $0 \le \phi \in C_0^\infty(\mathbf{R}^n)$. Then direct calculation shows

$$\begin{aligned} &\int \phi h_j(x,s)|_0^t dx = \\ &\int_0^t \int \frac{u_j \Delta u_j}{\sqrt{u_j^2+\epsilon}}(x,t)\phi(x)dxds + \int_0^t \int \phi(x)\frac{u_j(x,s)}{\sqrt{u_j^2+\epsilon}} \int \rho_j(x-y)V(y,t)u(y,t)dydxdt \\ &\equiv T_1 + T_2. \end{aligned}$$

Using integration by parts, we deduce

$$\begin{aligned} T_1 = &- \int_0^t \int \frac{|\nabla u_j|^2}{\sqrt{u_j^2+\epsilon}}\phi(x,s)dxds + \int_0^t \int \frac{u_j^2}{u_j^2+\epsilon}\frac{|\nabla u_j|^2}{\sqrt{u_j^2+\epsilon}}\phi(x,s)dxds \\ &- \int_0^t \int u_j \frac{\nabla u_j \nabla \phi}{\sqrt{u_j^2+\epsilon}}dxds \end{aligned}$$

Since the sum of the first two terms on the righthand side of the above inequality is non-positive, we have

$$T_1 \le \int_0^t \int u_j \frac{|\nabla u_j| \ |\nabla \phi|}{\sqrt{u_j^2+\epsilon}} dxds.$$

Taking ϵ to zero, we obtain

$$\begin{aligned}&\int \phi |u_j(x,t)| dx \\ &\qquad \le \int_0^t \int |\nabla u_j| |\nabla \phi| dxds + \int_0^t \int \phi(x) |\int \rho_j V^+ u(y,s) dy| dxds \\ &\qquad\quad - \int_0^t \int \phi(x) \frac{u_j}{|u_j|}(x,s) \int \rho_j(x-y) V^- u(y,s) dydxds.\end{aligned}$$

Here and later we set $\frac{u_j}{|u_j|}(x,s)=0$ if $u_j(x,s)=0$.

Next, since $u, V \in L^2(\mathbf{R}^n \times (0,T))$, one has $uV \in L^1(\mathbf{R}^n \times (0,T))$. From the equation

$$\Delta u + Vu - \partial_t u = 0$$

one deduces

$$u(x,t) = \int_0^t \int G_0(x,t;y,s)(Vu)(y,s) dyds.$$

Here, as always, G_0 is the fundamental solution of the free heat equation. Hence $u(\cdot,t) \in L^1(\mathbf{R}^n)$ and $u \in L^1(\mathbf{R}^n \times (0,T))$. Therefore, for any fixed j, there holds

$$|\nabla u_j| \in L^1(\mathbf{R}^n \times (0,T)).$$

Now, for each $R>0$, we choose ϕ so that $\phi=1$ in $B(0,R)$, $\phi=0$ in $B(0,R+1)^c$ and $|\nabla \phi| \le 2$. Observing

$$\int_0^t \int |\nabla u_j| \ |\nabla \phi| dxds \to 0, \qquad R \to \infty,$$

we deduce, by letting $R \to \infty$,

(2.2)

$$\int |u_j(x,t)| dx \le \int_0^t \int |\int \rho_j V^+ u(y,s) dy| dxds - \int_0^t \int \frac{u_j}{|u_j|}(x,s) \int \rho_j(x-y) V^- u(y,s) dydxds.$$

By the fact that $V^-u, V^+u \in L^1(\mathbf{R}^n \times (0,T))$, we know that

$$\int \rho_j(\cdot - y) V^- u(y,\cdot) dy \to V^- u(\cdot,\cdot),$$

$$\int \rho_j(\cdot - y) V^+ u(y,\cdot) dy \to V^+ u(\cdot,\cdot),$$

in $L^1(\mathbf{R}^n \times (0,T))$. Since $\frac{u_j}{|u_j|}(x,s)$ is bounded and converges to $\frac{u}{|u|}(x,s)$ a.e. in the support of u, we have

$$\begin{aligned}&\Big| \int_0^t \int [\frac{u_j}{|u_j|}(x,s) \int \rho_j(x-y) V^- u(y,s) dy - \frac{V^- u^2}{|u|}] dxds \Big| \\ &\le \Big| \int_0^t \int (\frac{u_j}{|u_j|} - \frac{u}{|u|}) V^- u dxds \Big| + \Big| \int_0^t \int \frac{u_j}{|u_j|} \big(\rho_j \star V^- u - V^- u\big) dxds \Big| \to 0.\end{aligned}$$

Substituting this to (2.2) we deduce, by taking $j \to \infty$,

$$\int |u(x,t)|dx \le \int_0^t \int V^+|u(x,s)|dxds - \int_0^t \int \frac{V^- u^2}{|u|}(x,s)dxds.$$

Therefore

$$\int |u(x,t)|dx \le \int_0^t \int |u(x,s)|dxds \; \|V^+\|_\infty.$$

By Grownwall's inequality $u(x,t) = 0$ a.e. This proves part (i).

Proof of (ii).

Notice that the only place we have used the L^2 boundedness of u is to ensure that $Vu \in L^1(\mathbf{R}^n \times (0,T))$. But this a part of the assumptions in (ii). Therefore (ii) is also proven.

Proof of (iii). The uniqueness is an immediate consequence of part (i). So we only need to prove existence. This follows from a standard limiting process. For completeness we sketch the proof.

Given $k = 1, 2, ..$ let V_k be the truncated potential

$$V_k = \sup\{V(x,t), -k\}.$$

Since V_k is a bounded function there exists a unique, nonnegative solution u_k to the following problem.

$$\begin{cases} \Delta u_k + V_k u_k - \partial_t u_k = 0, \quad \text{in} \quad \mathbf{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) \ge 0, \quad x \in \mathbf{R}^n, \quad u_0 \in L^2(\mathbf{R}^n). \end{cases}$$

By the standard maximum principle, $\{u_k\}$ is a nonincreasing sequence and

$$\frac{1}{2}\int u_k^2|_0^T dx + \int_0^T \int |\nabla u_k|^2 dxdt = \int_0^T \int v_k u_k^2 dxdt \le \int_0^T \int V_k^+ u_k^2 dxdt \le \|V^+\|_\infty \int_0^T \int u_k^2 dxdt.$$

By Grownwall's lemma, we have

$$\int_0^T \int |\nabla u_k|^2 dxdt + \int u_k^2(x,T)dx \le \int u_0^2(x)dx + \|v^+\|_\infty \int u_0^2(x)dx \; e^{2\|v^+\|_\infty T}.$$

It follows that u_k converges pointwise to a function u which also satisfies the above inequality. Let ϕ be a test function with compact support. Then

$$\int (u_k\phi)|_0^T dx - \int_0^T \int u_k \phi_t dxdt - \int_0^T \int u_k \Delta\phi dxdt - \int_0^T \int V_k u_k \phi dxdt = 0$$

Since $\{u_k\}$ is a monotone sequence and also since $|V_k u_k| \le |V|u_1 \in L^1(\mathbf{R}^n \times (0,T))$, the dominated convergence theorem implies that

$$\int (u\phi)|_0^T dx - \int_0^T \int u \phi_t dxdt - \int_0^T \int u \Delta\phi dxdt - \int_0^T \int V u \phi dxdt = 0.$$

This shows that u is a nonnegative solution. It is clear that u is not identically zero since u_0 is not. This proves part (iii) of the proposition.

Proof of Part (iv).

Let V_k be a truncated potential as in part (iii). Since V_k is bounded, the standard maximum principle shows that there exists a unique, nonnegative solution to the following problem.

$$\begin{cases} \Delta u_k + V_k u_k - \partial_t u_k = f \le 0, \quad \mathbf{R}^n \times (0,T), \ T>0, \\ u_k(\cdot,0)=0. \end{cases}$$

Moreover $\{u_k\}$ forms a decreasing sequence. Since V_k is a bounded function, the standard parabolic theory shows that

$$\int u_k(x,t)dx = \int_0^t \int V_k^+ u_k dxds - \int_0^t \int V_k^- u_k dxds + \int_0^t \int f dxds.$$

Therefore

$$\int u_k(x,t)dx \le \|V^+\|_\infty \int_0^t \int u_k dxds + \int_0^t \int f dxds.$$

This implies

$$\int u_k(x,t)dx + \int_0^t \int u_k dxds \le C(t, \|V^+\|_\infty, \|f\|_1).$$

It follows that

$$\int u_k(x,t)dx + \int_0^t \int u_k dxds + \int_0^t \int V_k^- u_k dxds \le C(t, \|V^+\|_\infty, \|f\|_1).$$

Let w be the pointwise limit of the decreasing sequence $\{u_k\}$. Then we have

$$\int w(x,t)dx + \int_0^t \int w dxds \le C(t, \|V^+\|_\infty, \|f\|_1).$$

It is straight forward to check that w is a nonnegative solution to the problem

$$\begin{cases} \Delta w + V w - \partial_t w = f \le 0, \quad \mathbf{R}^n \times (0,T), \ T>0, \\ w(\cdot,0)=0. \end{cases}$$

Hence

$$\begin{cases} \Delta (w-u) + V (w-u) - \partial_t (w-u) = 0, \quad \mathbf{R}^n \times (0,T), \ T>0, \\ (w-u)(\cdot,0)=0. \end{cases}$$

Recall that u is assumed to be a L^2 solution and that $V \in L^2$. We have that $Vu \in L^1$ and consequently $u(\cdot,t) \in L^1(\mathbf{R}^n)$ and $u \in L^1(\mathbf{R}^n \times (0,T))$. Now by Part (ii) of the proposition, we deduce $w=u$ since w is also L^1. Hence $u \ge 0$. This finishes the proof of the proposition. □

2.3. **Proofs of Theorems.**

Proof of Theorem 2.1 (i).
For $j = 1, 2...$, let

$$V_j = \min\{V(x,t), j\}.$$

Since V_j is bounded from above, Proposition 2.1 shows that there exists a unique solution u_j to the following problem.

$$\begin{cases} \Delta u_j + V_j u_j - \partial_t u_j = 0, \qquad (x,t) \in \mathbf{R}^n \times (0,\infty) \\ u_j(x,0) = u_0(x), \qquad x \in \mathbf{R}^n. \end{cases}$$

Notice that $u_j - u_{j-1}$ is a solution to the problem

$$\begin{cases} \Delta(u_j - u_{j-1}) + V_j(u_j - u_{j-1}) - \partial_t(u_j - u_{j-1}) = (V_{j-1} - V_j)u_{j-1}, & (x,t) \in \mathbf{R}^n \times (0,\infty) \\ (u_j - u_{j-1})(x,0) = 0, \qquad x \in \mathbf{R}^n. \end{cases}$$

Notice that

$$(V_{j-1} - V_j)u_{j-1} \leq 0, \qquad (V_{j-1} - V_j)u_{j-1} \in L^1(\mathbf{R}^n \times (0,T)), \quad T > 0.$$

We can then apply Proposition 2.1 (iv) to conclude that

$$u_j \geq u_{j-1}.$$

Moreover

$$\begin{cases} \Delta(u - u_j) + V_j(u - u_j) - \partial_t(u - u_j) = (V_j - V)u, & (x,t) \in \mathbf{R}^n \times (0,\infty) \\ (u - u_j)(x,0) = 0, \qquad x \in \mathbf{R}^n, \end{cases}$$

with

$$(V_j - V)u \leq 0, \qquad (V_j - V)u \in L^1(\mathbf{R}^n \times (0,T)), \quad T > 0.$$

By Proposition 2.1 (iv) again we know that

$$u \geq u_j.$$

Therefore $\{u_j\}$ is a non-decreasing sequence of nonnegative functions bounded from above by a L^2 function. Let w be the pointwise limit of u_j. The w is L^2 and $|V_j u_j| \leq |Vu| \in L^1(\mathbf{R}^n \times (0,T))$, $T > 0$. By the dominated convergence theorem, it is straight forward to check that w is a nonnegative L^2 solution to the equation

$$\begin{cases} \Delta w + Vw - \partial_t w = 0, & (x,t) \in \mathbf{R}^n \times (0,\infty) \\ w(x,0) = u_0(x), \qquad x \in \mathbf{R}^n. \end{cases}$$

Fixing j, for any $k = 1, 2, ...$, Let

$$V_{jk} = \max\{V_j(x,t), -k\}.$$

Since V_{jk} is bounded, the following problem has a unique L^2 solution.

$$\begin{cases} \Delta u_{jk} + V_{jk}u_{jk} - \partial_t u_{jk} = 0, & (x,t) \in \mathbf{R}^n \times (0,\infty) \\ u_j(x,0) = u_0(x), \qquad x \in \mathbf{R}^n. \end{cases}$$

Due to the fact that $\{V_{jk}\}$ is a decreasing sequence of k, the maximum principle shows that $\{u_{jk}\}$ is also a decreasing sequence of k. Since $V_{jk}u_{jk} \in L^2(\mathbf{R}^n \times (0,T))$, $T > 0$, the parabolic version of the Calderon-Zygmond theory shows

$$\Delta u_{jk}, \quad \partial_t u_{jk} \in L^2(\mathbf{R}^n \times (0,T)), T > 0.$$

Since

$$0 \leq u_{jk} - u_j \leq u_{j1} - u_j, \ k = 1, 2, 3, ...,$$
$$0 \leq w - u_j \leq w - u_1,$$

we can apply the dominated convergence theorem to conclude that

$$\lim_{k \to \infty} \int_0^T \int (u_{jk} - u_j)^2 dxdt = 0, \quad \lim_{j \to \infty} \int_0^T \int (w - u_j)^2 dxdt = 0.$$

Therefore we can extract a subsequence $\{u_{jk_j}\}$ such that

$$\lim_{j\to\infty}\int_0^T\int (w-u_{jk_j})^2 dxdt = 0.$$

Hence there exists a subsequence, still called $\{u_{jk_j}\}$ such that

$$u_{jk_j} \to w \qquad a.e.$$

Recall that $u_0 \ge 0$, $u_0 \ne 0$ and V_{jk_j} is bounded. It is clear that $u_{jk_j} > 0$ when $t > 0$. Now we define

$$f_j = -\ln u_{jk_j}, \qquad f = -\log w.$$

Then

$$V_{jk_j} = \Delta f_j - |\nabla f_j|^2 - \partial_t f_j.$$

Clearly $f_j \to f$ a.e. and $V_{jk_j} \to V$ in L^2 as $j \to \infty$. By Definition 2.1, this means

$$V = \Delta f - |\nabla f|^2 - \partial_t f.$$

It is clear that $e^{-f}b = w$ is L^2 by construction.

Proof of Theorem 2.1 (ii).

By assumption, there exist sequences of functions $\{V_j\}$ and $\{f_j\}$ such that

$$\|V_j\|_{L^2} \le C, \quad |V_j| \le |V_{j+1}|, \quad f_j \in L^\infty, \quad \|V_j - V\|_{L^2} \to \infty,$$
$$V_j = \Delta f_j - |\nabla f_j|^2 - \partial_t f_j.$$

Then for $u_j \equiv e^{-f_j}$, we have

$$\Delta u_j + V_j u_j - \partial_t u_j = 0.$$

We will show that $\|u_j\|_{L^2}$ is uniformly bounded. To this end, we observe that

$$\Delta(u_j - u_{j+1}) + V_j(u_j - u_{j+1}) - \partial_t(u_j - u_{j+1}) = -(V_j - V_{j+1})u_{j+1} \ge 0.$$

Recall from Definition 2.1 that $f_j(x,0)$ is independent of j. Hence $u_j(x,0) = u_{j+1}(x,0)$. Therefore $0 \le u_j \le u_{j+1}$. By the assumption that $f_j \to f$ a.e., we know that $u_j = e^{-f_j} \to e^{-f}$ a.e. Note that $e^{-f} \in L^2(\mathbf{R}^n \times (0,\infty))$. Hence $\|u_j\|_{L^2}$ is uniformly bounded.

By weak compactness in L^2, there exists a subsequence, still called $\{u_j\}$ such that u_j converges weakly to a L^2 function which we will call u. Observe that, for any compactly supported test function ϕ, there holds

$$\begin{aligned}\|u_jV_j\phi - uV\phi\|_{L^1} &\le \|u_j(V_j - V)\phi\|_{L^1} + \|(u_j-u)V\phi\|_{L^1}\\ &\le \|u_j\|_{L^2}\|V_j - V\|_{L^2}\ \|\phi\|_{L^\infty} + \|(u_j-u)V\phi\|_{L^1}.\end{aligned}$$

Hence

$$\|u_jV_j\phi - uV\phi\|_{L^1} \to 0$$

when $j \to \infty$. From here it is easy to check that u is a nonnegative solution to (1.1) with $u_0 = e^{-f_j(x,0)}$ as the initial value. Note the u_0 is independent of j. If $u_0 \in L^2(\mathbf{R}^n)$ then we are done. Otherwise, we can selection a L^2 function dominated by u_0 to serve as the initial value. □

Next we will provide a

Proof of Theorem 2.2 (i).

We will use an idea based on an argument in [St] where the heat equation with some singular, time independent potentials are studied.

By virtue of Proposition 2.1, the fundamental solution G_V is defined as the limit of fundamental solutions of the equation in (1.1) where V is replaced by nonsingular potentials. Therefore we can and will assume that V is smooth in this subsection. The constants involved will be independent of the smoothness.

Since, by assumption

$$V = \Delta f - \alpha|\nabla f|^2 - \partial_t f, \tag{2.3}$$

one has

$$\alpha V = \Delta(\alpha f) - |\nabla(\alpha f)|^2 - \partial_t(\alpha f).$$

Writing $F = e^{-\alpha f}$, it is easy to show that

$$\Delta F + \alpha V F - \partial_t F = 0. \tag{2.4}$$

Let us denote the fundamental solution of the equation in (2.4) by $G_{\alpha V}$. Since f is bounded, we know that F is bounded between two positive constants. Therefore it is clear that

$$0 < \frac{\inf F}{\sup F} \le \int G_{\alpha V}(x, t; y, s) dy \le \frac{\sup F}{\inf F}, \tag{2.5}$$

for all $x \in \mathbf{R}^n$ and $t > s$. Here $\inf F$ and $\sup F$ are taken over the whole domain of F.

By Feynman-Kac formula and Hölder's inequality, for a given $\phi \in C_0^\infty(\mathbf{R}^n)$, there holds

$$\Big|\int G_V(x, t; y, s)\phi(y)dy\Big| \le \Big[\int G_{\alpha V}(x, t; y, s)dy\Big]^{1/\alpha} \ \Big[\int G_0(x, t; y, s)|\phi(y)|^{\alpha/(\alpha-1)}dy\Big]^{(\alpha-1)/\alpha}$$

By (2.5), we deduce

$$\Big|\int G_V(x, t; y, s)\phi(y)dy\Big| \le \frac{c s_0^{1/\alpha}}{(t-s)^{(\alpha-1)n/(2\alpha)}}\|\phi\|_{\alpha/(\alpha-1)},$$

where $s_0 = \frac{\sup F}{inf F}$. The norm on ϕ means the $L^{\alpha/(\alpha-1)}(\mathbf{R}^n)$ norm. Hence

$$\|G_V(\cdot, t; \cdot, s)\|_{\alpha/(\alpha-1),\infty} \le \frac{c s_0^{1/\alpha}}{(t-s)^{(\alpha-1)n/(2\alpha)}}. \tag{2.6}$$

Here and later the norm $\|\cdot\|_{p,q}$ stands for the operator norm from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ for p, q between 1 and ∞.

Without loss of generality we assume that $\alpha/(\alpha-1)$ is an integer. This is so because otherwise we can choose one $\alpha_1 \in (1, \alpha)$ such that $\alpha_1/(\alpha_1 - 1)$ is an integer. Then interpolating between $G_{\alpha V}$ and G_0 by Feynman-Kac formula again, we know that

$$\int G_{\alpha_1 V}(x, t; y, s)dy \le C(F, \alpha, \alpha_1).$$

Then we can just work with $G_{\alpha_1 V}$ instead of $G_{\alpha V}$ in the above.

Using the reproducing property of G_V we deduce

$$\|G_V(\cdot, t; \cdot, s)\|_{1,\infty} \le \Pi_{j=1}^m \|G_V(\cdot, s + t_j;\ \cdot, t_{j-1})\|_{p_j,q_j}, \tag{2.7}$$

where

$$m = \alpha/(\alpha-1), \quad p_j = m/(m-l+1), \quad q_j = m/(m-l), \quad t_j = s + ((t-s)j/m).$$

For each j between 1 and m, we apply the Riesz-Thorin interpolation theorem to deduce

$$\|G_V(\cdot, t_j;\ \cdot, t_{j-1})\|_{p_j,q_j} \le \|G_V(\cdot, s + t_j;\ \cdot, t_{j-1})\|_{1,m/(m-1)}^{1-\lambda_j} \ \|G_V(\cdot, t_j;\ \cdot, t_{j-1})\|_{m,\infty}^{\lambda_j}.$$

Here the parameters are determined by the following relations

$$\frac{1}{p_j} = \frac{1-\lambda_j}{1} + \frac{\lambda_j}{m}, \qquad \frac{1}{q_j} = \frac{1-\lambda_j}{m/(m-1)} + \frac{\lambda_j}{\infty},$$
$$\lambda_j = \frac{j-1}{m-1}.$$

It follows that

$$\|G_V(\cdot, t_j;\ \cdot, t_{j-1})\|_{p_j,q_j} \le \|G_V(\cdot, t_j;\ \cdot, t_{j-1})\|_{m,\infty}. \tag{2.8}$$

Substituting (2.6) to (2.8), we deduce, after noticing that $t_j - t_{j-1} = (t-s)/m$,

$$\|G_V(\cdot, t_j;\ \cdot, t_{j-1})\|_{p_j,q_j} \le \frac{c s_0^{1/\alpha} m^{(\alpha-1)n/(2\alpha)}}{(t-s)^{(\alpha-1)n/(2\alpha)}}.$$

This and (2.7) imply that

$$\|G_V(\cdot, t; \cdot, s)\|_{1,\infty} \le \frac{c s_0^{m/\alpha} m^{n/2}}{(t-s)^{n/2}}.$$

Here we just used the relation $m = \alpha/(\alpha-1)$. This yields the on-diagonal upper bound

$$G_V(x, t; y, s) \le \frac{c s_0^{1/(\alpha-1)} (\alpha/(\alpha-1))^{n/2}}{(t-s)^{n/2}}. \tag{2.9}$$

In order to obtain the full Gaussian bound, we observe that, for any $p > 1$, the Feynman-Kac formula implies

$$G_V(x, t; y, s) \le \left[G_{pV}(x, t; y, s)\right]^{1/p} \left[G_0(x, t; y, s)\right]^{(p-1)/p}. \tag{2.10}$$

Notice also

$$pV = p(\Delta f - \alpha |\nabla f|^2 - \partial_t f) = \Delta(pf) - \frac{\alpha}{p}|\nabla(pf)|^2 - \partial_t(pf).$$

Taking $p = (1+\alpha)/2$, then $\frac{\alpha}{p} > 1$. Therefore pV also satisfies the condition of Theorem 2.2 (i). Hence, the on-diagonal bound (2.9) holds for G_{pV}. i.e., there exists a constant $C(\alpha, e^{\sup f - \inf f})$ such that

$$G_{pV}(x, t; y, s) \le \frac{C(\alpha, e^{\sup f - \inf f})}{(t-s)^{n/2}}.$$

Substituting this to the inequality (2.10), we obtain the desired Gaussian upper bound for G_V.

Proof of (ii).

In this part we prove the Gaussian lower bound. We will follow Nash's original idea. The novelty is a way of handling the potential term even if it is very singular. The main idea is to exploit the structure of the potential when it is written as a combination of derivatives.

Since the setting of our problem is invariant under the scaling, for $r > 0$,

$$V_r(x, t) = r^2 V(rx, r^2 t), \qquad f_r(x, t) = f(rx, r^2 t), \qquad u_r(x, t) = r^2 u(rx, r^2 t),$$

we can just prove the lower bound for $t = 1$ and $s = 0$. We divide the proof into three steps.

Step 1. Fixing $x \in \mathbf{R}^n$, let us set

$$u(y, s) = G_V(y, s; x, 0),$$

$$H(s) = \int e^{-\pi|y|^2} \ln u(y,s) dy.$$

Differentiating $H(s)$, one obtains

$$\begin{aligned} H'(s) &= \int e^{-\pi|y|^2} \frac{\partial_s u(y,s)}{u(y,s)} dy \\ &= -\int \nabla \Big(\frac{e^{-\pi|y|^2}}{u} \Big) \nabla u dy + \int e^{-\pi|y|^2} V(y,s) dy. \end{aligned}$$

Estimating the first term on the righthand side of the above inequality as in [FS], section 2, one arrives at

$$H'(s) \ge -C + \frac{1}{2} \int e^{-\pi|y|^2} |\nabla \ln u(y,s)|^2 dy + \int e^{-\pi|y|^2} V(y,s) dy. \tag{2.11}$$

Here C is a positive constant. Since,

$$V = \Delta f - \alpha |\nabla f|^2 - \partial_t f,$$

we know that

$$\begin{aligned} f(x,t) = \int & G_0(x,t;y,0) f(y,0) \\ & - \int_0^t \int G_0(x,t;y,s) V(y,s) dy ds - \alpha \int_0^t \int G_0(x,t;y,s) |\nabla f|^2 dy ds. \end{aligned}$$

By our assumption

$$\int_0^t \int G_0(x,t;y,s) |\nabla f|^2 dy ds < \infty.$$

Hence the boundedness of f implies that

$$m(x,t) \equiv -\int_0^t \int G_0(x,t;y,s) V(y,s) dy ds \in L^\infty.$$

Moreover

$$\Delta m - \partial_t m = V. \tag{2.12}$$

Therefore

$$\begin{aligned} \int e^{-\pi|y|^2} V(y,s) dy &= \int e^{-\pi|y|^2} [\Delta m - \partial_s m](y,s) dy \\ &= \int [\Delta e^{-\pi|y|^2}] m dy - \partial_s \int e^{-\pi|y|^2} m(y,s) dy \\ &\ge -C - \partial_s \int e^{-\pi|y|^2} m(y,s) dy. \end{aligned}$$

Here we have used the boundedness of m. Substituting the above to the righthand side of (2.11), we obtain

$$H'(s) \ge -C + \frac{1}{2} \int e^{-\pi|y|^2} |\nabla \ln u(y,s)|^2 dy - M'(s), \tag{2.13}$$

where

$$M(s) = \int e^{-\pi|y|^2} m(y,s) dy. \tag{2.14}$$

Step 2. By Poincaré's inequality with $e^{-\pi|y|^2}$ as weight, we deduce, for some $B > 0$,

$$H'(s) \geq -C + B \int e^{-\pi|y|^2} [\ln u(y,s) - H(s)]^2 dy - M'(s).$$

Next, observe that $(\ln u - H(s))^2/u$ is non-increasing as a function of u when u is between $e^{2+H(s)}$ and ∞. Also from the Gaussian upper bound,

$$\sup_{1/2 \leq s \leq 1} u(y,s) \leq K < \infty.$$

Therefore

$$H'(s) \geq -C + CBK^{-1}(\ln K - H(s))^2 \int_{u(y,s) \geq \exp(2+H(s))} e^{-\pi|y|^2} u(y,s) dy - M'(s). \tag{2.15}$$

Using the Gaussian upper bound again, we know that $H(s) \leq C$ for some $C > 0$ and that

$$\begin{aligned}
\int_{u(y,s) \geq exp(2+H(s))} e^{-\pi|y|^2} u(y,s) dy &\geq \int e^{-\pi|y|^2} u(y,s) dy - ce^{2+H(s)} \\
&\geq e^{-\pi r^2} \int_{|y|<r} u(y,s) dy - ce^{2+H(s)} \\
&= e^{-\pi r^2} \Big[\int u(y,s) dy - \int_{|y|<r} u(y,s) dy \Big] - ce^{2+H(s)}.
\end{aligned} \tag{2.16}$$

We aim to find a lower bound for the righthand side of (2.16). By (2.12),

$$V = \Delta m - \partial_t m \geq \Delta m - |\nabla m|^2 - \partial_t m \equiv V_1.$$

Write $h = e^{-m}$. Then

$$\Delta h + V_1 h - \partial_t u = 0.$$

Since $m \in L^\infty$, we know that h is bounded between two positive constants. Observe that

$$h(x,t) = \int G_{V_1}(x,t;y,s) h(y,s) dy.$$

Hence

$$0 < c_1 < \int G_{V_1}(x,t;y,s) dy \leq c_2.$$

By the maximum principle, we have

$$\int G_V(x,t;y,s) dy \geq \int G_{V_1}(x,t;y,s) dy \geq c_1 > 0. \tag{2.17}$$

Recall that $u(y,s) = G_V(y,s;x,0)$. Substituting (2.17) to (2.16) and applying the Gaussian upper bound on $u(y,s)$, we deduce

$$\int_{u(y,s) \geq exp(2+H(s))} e^{-\pi|y|^2} u(y,s) dy \geq ce^{-\pi r^2} c_1 - ce^{2+H(s)}, \tag{2.18}$$

when r is sufficiently large. Substituting (2.18) to (2.15), we arrive at

$$H'(s) \geq -C + CBK^{-1}(\ln K - H(s))^2 e^{-\pi r^2} c_1 - ce^{2+H(s)} - M'(s). \tag{2.19}$$

We claim that $H(1) \geq -c_0$ for some sufficiently large $c_0 > 0$. Suppose otherwise, i.e. $H(1) < -c_0$. From (2.19), for some $C > 0$,

$$H'(s) \geq -C - M'(s).$$

Hence

$$H(1) - H(s) \geq -C(1-s) - (M(1) - M(s)).$$

Therefore

$$H(s) \leq H(1) + C(1-s) + (M(1) - M(s)) \leq -c_0/2,$$

when c_0 is chosen sufficiently large. It follows from (2.19) that

$$H'(s) \geq -c_1 + c_2 H^2(s) - M'(s).$$

This shows

$$(H(s) + M(s))' \geq -c_3 + c_4 (H(s) + M(s))^2.$$

From here, one immediately deduces

$$H(1) \geq -A, \qquad A > 0.$$

The claim is proven. Thus

$$\int e^{-\pi|y|^2} \ln G_V(y, s; x, 0) dy \geq -c_0$$

where $|x - y| \leq 1$.

Using the reproducing property of G_V and Jensen's inequality, we have, when $|x-y| \leq 1$,

$$\begin{aligned} \ln G_V(x, 2; y, 0) &= \ln \int G_V(x, 2; z, 1) G_V(z, 1; y, 0) dz \\ &\geq \int e^{-\pi|y|^2} \ln G_V(x, 2; z, 1) dz + \int e^{-\pi|y|^2} \ln G_V(z, 1; y, 0) dz \geq -C. \end{aligned}$$

This proves the on-diagonal lower bound. The full Gaussian lower bound now follows from the standard argument in [FS]. □

Proof of (iii).

Since,

$$V = \Delta f - \alpha |\nabla f|^2 - \partial_t f,$$

we have

$$V + (\alpha - 1)|\nabla f|^2 = \Delta f - |\nabla f|^2 - \partial_t f.$$

Let $u = e^{-f}$, by direct calculation,

$$\Delta u + Vu - \partial_t u + (\alpha - 1)|\nabla f|^2 u = 0.$$

Hence

$$u(x, t) = \int G_V(x, t; y, 0) u(x, 0) dy + (\alpha - 1) \int_0^t \int G_V(x, t; y, s) |\nabla f|^2(y, s) u(y, s) dy ds.$$

Since $f \in L^\infty$, we know that u is bounded between two positive constants. If, by assumption, G_V has a Gaussian lower bound, then, for some $b > 0$, we have

$$\int_0^t \int g_b(x, t; y, s) |\nabla f|^2(y, s) dy ds \leq C \frac{\sup u}{\inf u}.$$

This completes the proof of part (iii) of Theorem 2.2. □

Proof of Theorem 2.3.

(i). We write

$$V_1 = 2(\Delta f - \partial_t f - 3|\nabla f|^2), \qquad V_2 = 6|\nabla f|^2.$$

Let G_{V_i}, $i=1,2$, be the fundamental solution of $\Delta u + V_i u - \partial_t u = 0$. Since

$$V = \Delta f - \partial_t f = (V_1/2) + (V_2/2),$$

the Feynman-Kac formula implies

$$G_V(x,t;y,s) \le \big[G_{V_1}(x,t;y,s)\big]^{1/2} \ \big[G_{V_2}(x,t;y,s)\big]^{1/2}.$$

Observe that

$$V_1 = \Delta(2f) - \partial_t(2f) - \frac{3}{2}|\nabla(2f)|^2.$$

Hence, by Theorem 2.2, we know that G_{V_1} has Gaussian upper bound. Under the smallness assumption on the $L^{p,q}$ norm of $|\nabla f|^2$ in the theorem, it is well known that G_{V_2} also has a Gaussian upper bound. Therefore G_V has Gaussian upper bound.

In order to prove the Gaussian lower bound, we observe that

$$V = \Delta f - \partial_t f \ge V_3 \equiv \Delta f - \partial_t f - 2|\nabla f|^2.$$

Under our assumption on the $L^{p,q}$ norm of $|\nabla f|$, it is straight forward to check that

$$g_{1/4} \star |\nabla f|^2 \in L^\infty.$$

Hence Theorem 2.2 (ii) shows G_{V_3} has Gaussian lower bound. Clearly this Gaussian lower bound of G_{V_3} is also a Gaussian lower bound of G_V by the maximum principle. This proves part (a).

(ii). Clearly we can choose A_0 sufficiently small so that all the following kernels have global Gaussian upper and lower bound:

$$G_{2V}, \quad G_{V/2}, \quad G_{2\Delta f}, \quad G_{\Delta f/2}. \tag{2.20}$$

The bounds on the first two kernels follow from part (i). The bounds on the last two kernels follow from standard theory since $\Delta f = div(\nabla f)$ with ∇f has a small norm in the suitable $L^{p,q}$ class. (see [LS] e.g.)

Now observe that

$$-\partial_t f = \Delta f - \partial_t f - \Delta f = V - \Delta f,$$
$$\frac{V}{2} = \frac{\Delta f}{2} - \frac{\partial_t f}{2}.$$

By Feynman-Kac formula

$$G_{(-\partial_t f)} \le \big(G_{2V}\big)^{1/2} \ \big(G_{2\Delta f}\big)^{1/2};$$

$$G_{V/2} \le \big(G_{2\Delta f}\big)^{1/2} \ \big(G_{-\partial_t f}\big)^{1/2}.$$

Hence (2.20) show that $G_{(-\partial_t f)}$ also has global Gaussian upper and lower bound. Since the setting of the Theorem is invariant under the reflection $f \to -f$ the result follows. $\square$

We close this section by giving proofs of the corollaries.

Proof of Corollary 1.

(a). Let $V_k = \min\{V(x,t), k\}$, $k = 1, 2, ...$ Then (1.1) with V replaced by V_k has a unique solution.

Let $J(t) \equiv \int_{\mathbf{R}^n} u_k^2(x,t)dx$. Then

$$J'(t) = 2\int_{\mathbf{R}^n} [-\nabla u_k \nabla u_k + V_k u_k^2]dx.$$

By our assumption on V,

$$J'(t) \le 2bJ(t)$$

which implies

$$\int_{\mathbf{R}^n} u_k^2(x,t)dx \le \int_{\mathbf{R}^n} u_0^2(x)dx \ e^{2bt}.$$

Therefore if $u_0 \in L^2(D)$, we conclude that $u_k(x,t)$ increases to a finite positive limit $u(x,t)$ as $k \to \infty$, for all t and for a.e. x. Moreover $u(\cdot,t) \in L^2(\mathbf{R}^n)$. We show that the above u is a positive L^2 solution to (1.1).

Since u_k is a solution to (1.1) with V replaced by V_k, for any $\psi \in C_0^\infty(\mathbf{R}^n \times (0,T))$, we have

$$\int (u_k\psi)|_{t_1}^{t_2} dx - \int_{t_1}^{t_2}\int u_k\psi_t dxdt - \int_{t_1}^{t_2}\int u_k\Delta\psi dxdt - \int_{t_1}^{t_2}\int V_k u_k\psi dxdt = 0$$

for all $t_1, t_2 \in (\delta, t_0)$.

By our assumption $|V_k u_k| \le |Vu| \in L^1(\mathbf{R}^n \times (0,T))$. Taking $k \to \infty$ and using the dominated convergence theorem, we obtain

$$\int (u\psi)|_{t_1}^{t_2} dx - \int_{t_1}^{t_2}\int u\psi_t dxdt - \int_{t_1}^{t_2}\int u\Delta\psi dxdt - \int_{t_1}^{t_2}\int Vu\psi dxdt = 0.$$

This shows that u is a positive solution to (1.1). By Theorem 2.1

$$V = \Delta f - |\nabla f|^2 - \partial_t f.$$

(b). Suppose

$$V = \Delta f - |\nabla f|^2.$$

Due to the L^2 convergence, it suffices to prove that V_j in Definition 2.1 satisfies (2.1). Let ϕ be a test function, then

$$\begin{aligned}\int_0^\infty\int V_j\phi^2 dxdt &= \int_0^\infty\int [\Delta f_j - |\nabla f_j|^2]\phi^2 dxdt\\ &= -2\int_0^\infty\int \nabla f_j\nabla\phi\ \phi dxdt - \int_0^\infty\int |\nabla f_j|^2\phi^2 dxdt\\ &\le \int_0^\infty\int |\nabla\phi|^2 dxdt\phi.\end{aligned}$$

(c). The statement is self-evident by part (b) and Theorem 2.1. □

Proof of Corollary 2.

Suppose $V = \Delta f - |\nabla f|^2 + b$. Then, by the same limiting argument as above, we have

$$\begin{aligned}\int V\phi^2 dx &= \lim_{j\to\infty}\int [\Delta f_j - |\nabla f_j|^2]\phi^2 dx + b\int \phi^2 dx\\ &= \lim_{j\to\infty}\Big(-2\int \nabla f_j\nabla\phi\ \phi dx - \int |\nabla f_j|^2\phi^2 dx\Big) + b\int\phi^2 dx.\end{aligned}$$

Therefore

$$\int V\phi^2 dx \le \int |\nabla\phi|^2 dx + b\int \phi^2 dx.$$

Also by part (a) of Corollary 1, (1.1) has a positive solution when $u_0 \ge 0$.

On the other hand, suppose V satisfies

$$\int V\phi^2 dx \le \int |\nabla \phi|^2 dx + b \int \phi^2 dx.$$

Write $V_j = \min\{V, j\}$ with $j = 1, 2, ...$ Then

$$\int (V_j - b)\phi^2 dx \le \int |\nabla \phi|^2 dx.$$

Notice that $V_j - b$ is a bounded function. Hence we can apply Theorem C.8.1 in [Si] to conclude that there exists $u_j > 0$ such that

$$\Delta u_j + (V_j - b)u_j = 0.$$

Writing $f_j = -\ln u_j$, we have

$$V_j = \Delta f_j - |\nabla f_j|^2 + b.$$

By definition, this means

$$V = \Delta f - |\nabla f|^2 + b.$$

□

3. HEAT BOUNDED FUNCTIONS AND THE HEAT EQUATION

Here we introduce another class of singular functions that has its origin in the Kato type class. As mentioned in the introduction, a function is in a Kato type class if the convolution of the absolute value of the function and the fundamental solution of Laplace or the heat equation is bounded. Here we generalize this notion by a simple but key stroke, i.e., we delete the absolute value sign on the function in the definition of the Kato class. More precisely, we have

Definition 3.1. Let $f = f(x,t)$ be a local L^1 function in space time and G_0 be the standard Gaussian in $\mathbf{R}^n$. We say that f is *heat bounded* in a domain $\Omega \subset \mathbf{R}^n \times \mathbf{R}^1$ if

$$G_0 \star f(x,t) \equiv \int_0^t \int_{\mathbf{R}^n} G_0(x,t;y,s) f(y,s) dy ds$$

is a bounded function in Ω.

We say that f is *almost heat bounded* in a domain $\Omega \subset \mathbf{R}^n \times \mathbf{R}^1$ if

$$G_0 \star f(x,t) \in L^p(\Omega)$$

for all $p > 1$.

Example. The function $V(x) = a\frac{\chi_{B(0,1)}}{|x|^2}$ is not heat bounded but is almost heat bounded in $\mathbf{R}^n$. Here a is a nonzero constant.

In the next two propositions, we provide a comparison between the heat bounded class and more familiar classes of functions.

Proposition 3.1. *Suppose, in the distribution sense, $V = \partial^2_{ij} f$ with $f \in \cap_{p>1} L^p(\mathbf{R}^n \times (0,T))$. Then V is almost heat bounded in $\mathbf{R}^n \times (0,T)$.*

Proof.

Let G_0 be the free heat kernel in $\mathbf{R}^n \times (0, \infty)$. By the assumption on f, the function $u = u(x,t)$, defined by

$$u(x,t) = \int_0^t \int_{\mathbf{R}^n} G_0(x,t;y) f(y,s) dyds$$

is a solution to the equation

$$\begin{cases} \Delta u(x,t) - u_t(x,t) = -f(x,t), & x \in \mathbf{R}^n, t > 0; \\ u(x,0) = 0. \end{cases}$$

By the parabolic version of the Calderon-Zygmond inequality (see [Lieb] e.g., we know that

$$u \in W^{2,p}(\mathbf{R}^n), \quad \forall p > 1.$$

Hence

$$\partial^2_{ij} \int_0^t \int_{\mathbf{R}^n} G_0(x,t;y) f(y,s) dyds \in L^p(\mathbf{R}^n), \quad \forall p > 1. \quad \square$$

Proposition 3.2. *Suppose, $0 \le V \in L^1_{loc}$ is form bounded in $D \times [0,T]$. i.e.*

$$\int_0^T \int_D V\phi^2 \le b_1 \int_0^T \int_D |\nabla \phi|^2 + b_2 \int_0^T \int_D \phi^2.$$

for all smooth, compactly supported function $\phi \in D \times [0,T] \subset \mathbf{R}^n \times [0,T]$. Then V is almost heat bounded in $D \times [0,T]$.

Proof.

We will only consider the case when $D = \mathbf{R}^n$. The other cases follow from the full space case by a standard comparison method.

Since one can consider cV with c sufficiently small otherwise, we can choose the constant b_1 in the definition of form boundedness to be $1/2$, i.e. we assume that

$$\int_0^T \int_D V\phi^2 \le \frac{1}{2} \int_0^T \int_D |\nabla \phi|^2 + b_2 \int_0^T \int_D \phi^2.$$

for all smooth, compactly supported function ϕ.

Let u_k be the solution of

$$\begin{cases} \Delta u_k + V_k u_k - (u_k)_t = 0, \quad \text{in} \quad \mathbf{R}^n \times (0, \infty), \quad V \in L^2_{loc}(\mathbf{R}^n \times (0,\infty)) \\ u_k(x,0) = u_0(x) > 0, \quad x \in \mathbf{R}^n, \quad u_0 \in L^2(\mathbf{R}^n). \end{cases} \tag{3.1}$$

Here V_k is the truncated potential $V_k = \min\{V, k\}$ with k being positive integers. Clearly $V_k \le V_{k+1}$.

We show that u_k converge pointwise to a locally integrable function.

Let $J(t) \equiv \int_D u_k^2(x,t)dx$. Then

$$J'(t) = 2 \int_D [-\nabla u_k \nabla u_k + V_k u_k^2] dx.$$

By our assumption on V,

$$J'(t) \le 2bJ(t)$$

which implies

$$\int_D u_k^2(x,t)dx \le \int_D u_0^2(x)dx \ e^{2bt}.$$

Therefore if $u_0 \in L^2(D)$, we conclude that $u_k(x,t)$ increases to a finite positive limit $u(x,t)$ as $k \to \infty$, for all t and for a.e. x. Moreover $u(\cdot,t) \in L^2(\mathbf{R}^n)$.

Write $w_k = \log u_k$. From (3.1), one deduces

$$\Delta w_k + |\nabla w_k|^2 + V_k - (w_k)_t = 0.$$

Therefore

$$\begin{aligned} w_k(x,t) = &\int_D G_0(x,t;y,0)w_k(x,0)dy \\ &+ \int_0^t \int_D G_0(x,t;y,s)|\nabla w_k(x,s)|^2 dyds + \int_0^t \int_D G_0(x,t;y,s)V_k(y,s)dyds. \end{aligned}$$

Therefore

$$\int_0^t \int_D G_0(x,t;y,s)V_k(y,s)dyds \le w_k(x,t) - \int_D G_0(x,t;y,0)w_k(x,0)dy.$$

By the monotone convergence theorem

$$\begin{aligned} \int_0^t \int_D G_0(x,t;y,s)V(y,s)dyds &\le w(x,t) - \int_D G_0(x,t;y,0)w(y,0)dy \\ &\le \log(1+u(x,t)) - \int_D G_0(x,t;y,0)\log u_0(y)dy. \end{aligned}$$

Now we take

$$u_0(x) = \frac{1}{1+|x|^n}.$$

Then, since $u_0 \in L^2(\mathbf{R}^n)$, we have

$$u(\cdot,t) \in L^2(\mathbf{R}^n).$$

By Jensen's inequality,

$$\log(1+u(\cdot,t)) \in L^p(\mathbf{R}^n), \qquad \forall p > 1.$$

It is also clear that

$$\int_D G_0(\cdot,t;y,0)\log(1+|y|^n)dy \in L^p_{loc}(\mathbf{R}^n), \qquad \forall p>1.$$

The result follows. □

4. Applications to the Navier-Stokes equation

In this section, we establish a new a priori estimate for a certain quantity involving the velocity and vorticity of the 3 dimensional Navier-Stokes equation.

$$\begin{aligned} &u_t - \Delta u(x,t) + u \cdot \nabla u(x,t) + \nabla p = 0, \\ &\nabla \cdot u = 0, \\ &u(x,0) = u_0(x) \end{aligned} \tag{4.1}$$

for $(x, t) \in \mathbb{R}^3 \times (0, \infty)$, where Δ is the standard Laplacian, a vector field u represents the velocity of the fluid, and a scalar field p the pressure. (The viscosity is normalized, $\nu = 1$.)

There has been an extensive and rapidly growing literature on the equation which is impossible to quote extensively here. Let us just mention that weak solutions are known to exist due to the seminal work of Leray [L]. However it is not known if the weak solution is smooth everywhere. Several sufficient conditions implying smoothness of weak solutions have been made. See for example [P] and [S]. In these two papers, it was shown that if the velocity u is in $L^{p,q}$ class with $\frac{3}{p} + \frac{2}{q} < 1$, then u is actually smooth. For more sufficiency results in various other spaces we refer the reader to the more recent survey paper [Ca]. However it is only known that $u \in L^{10/3,10/3}$. Therefore there is a gap in between the a priori estimate and the sufficiency condition.

What we will prove here is a different sufficiency condition and a priori estimate using the heat bounded and almost heat bounded potentials defined in the previous section. There is still a gap between the two conditions. However the gap seems logarithmic. More precisely, we have

Theorem 4.1. *Let u be a Leray-Hopf solution of the Navier-Stokes equation, which is classical in $\mathbf{R}^3 \times (0, T)$. Let w be the vorticity $\nabla \times u$. Define the quantity*

$$\mathbf{Q} = \mathbf{Q}(x,t) \equiv \frac{curl(u \times w) \cdot w + 2|\nabla \sqrt{|w|^2+1}|^2 - |\nabla w|^2}{|w|^2+1}(x,t).$$

Then the following statements hold for any $\delta \in (0, T)$.

(1). The quantity $\mathbf{Q}$ is almost heat bounded in $(\mathbf{R}^3 \times (\delta, T]) \cap \{|w| \ge 1\}$.

(2). u is a classical solution of the Navier-Stokes equation in $\mathbf{R}^3 \times (\delta, T]$ if and only if $\mathbf{Q}$ is heat bounded in $(\mathbf{R}^3 \times (\delta, T]) \cap \{|w| \ge 1\}$.

Remark 4.1.

The quantity Q is well defined since we assume that u is smooth for $t \in (0, T)$. The first term in Q is essentially the vortex stretching factor which is the hardest to control. The point of the theorem is that if there is blow up at time T, then the blow up just happens barely.

Proof of Theorem 4.1.

We will just prove (1) since (2) is self-evident afterward.

We divide the proof into three steps.

Step 1. rewriting the vortex equation in the log form.

Let $w = w(x, t)$ be the vortex. It is well known that $|w|^2$ satisfies the following scalar heat equation with lower order terms

$$\Delta |w|^2 - u \cdot \nabla |w|^2 + 2\alpha |w|^2 - 2|\nabla w|^2 - (|w|^2)_t = 0. \tag{4.2}$$

Here α is the vortex stretching potential given by (c.f. [Co])

$$\begin{aligned} \alpha(x,t) &= \frac{3}{4\pi} P.V. \int_{\mathbb{R}^3} D[\widetilde{y}, \widetilde{\omega}(x+y), \widetilde{\omega}(x)] \ |\omega(x+y,t)| \frac{dy}{|y|^3} \\ &= \frac{w \nabla u \cdot w}{|w|^2}. \end{aligned} \tag{4.3}$$

A straightforward computation from (4.2) shows

$$\Delta \ln(|w|^2+1) - u\cdot\nabla \ln(|w|^2+1) + 2\frac{\alpha|w|^2}{|w|^2+1} - 2\frac{|\nabla w|^2}{|w|^2+1} + \frac{|(\nabla|w|^2)|^2}{(|w|^2+1)^2} - \partial_t(\ln(|w|^2+1)) = 0.$$

Write $f = \frac{1}{2}\ln(|w|^2+1)$. We deduce

$$\Delta f - u\cdot\nabla f + \frac{\alpha|w|^2}{|w|^2+1} + 2|\nabla f|^2 - \frac{|\nabla w|^2}{|w|^2+1} - f_t = 0. \tag{4.4}$$

Step 2. a representation formula.

By our assumption, for $t \in (0,T)$, u and w are classical functions and f vanishes near infinity. This shows,

(4.5)

$$\begin{aligned} f(x,t) = &\int G_0(x,t;y,s) f_0(y)dy \\ &+ \int_0^t \int G_0(x,t;y,s)\left[\frac{\alpha|w|^2}{|w|^2+1} - u\cdot\nabla f + 2|\nabla f|^2 - \frac{|\nabla w|^2}{|w|^2+1}\right](y,s)dyds. \end{aligned}$$

Step 3. Apply Jensen's inequality.

For convenience, we write

$$Q \equiv \frac{\alpha|w|^2}{|w|^2+1} - u\cdot\nabla f + 2|\nabla f|^2 - \frac{|\nabla w|^2}{|w|^2+1} \tag{4.6}$$

It is clear that

$$\begin{aligned} Q &= \frac{w\nabla u\cdot w}{|w|^2+1} - \frac{1}{2}u_j\partial_j \ln(|w|^2+1) + 2|\nabla f|^2 - \frac{|\nabla w|^2}{|w|^2+1} \\ &= \frac{u\nabla w\cdot w}{|w|^2} - \frac{u_i\partial_j w_i w_i}{|w|^2+1} + 2|\nabla f|^2 - \frac{|\nabla w|^2}{|w|^2+1} \\ &= \frac{w\nabla u\cdot w - u\nabla w\cdot w + 2|\nabla\sqrt{|w|^2+1}|^2 - |\nabla w|^2}{|w|^2+1}. \end{aligned}$$

Following the well known vector identity, we have

$$Q = \frac{curl(u\times w)\cdot w + 2|\nabla\sqrt{|w|^2+1}|^2 - |\nabla w|^2}{|w|^2+1}. \tag{4.7}$$

It is well known that $w \in L^2(\mathbf{R}^3\times\mathbf{R}^+)$. Using Jensen's inequality, it is easy to show that $f(\cdot,t) \in L^p$ for any $p > 1$, in the region where $|w| \geq 1$. Hence the quantity Q is almost heat bounded. □

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